2 - 2

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Parrondo's Paradox 4

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6 October 2013

Introduction

In Parts 1 & 3 of this series of notebooks I took advantage of the finitistic means afforded when Parrondo's Game is modeled as a random walk on a cyclic graph of order 3. There long-term winning/losing was signaled by the sign of the probability current, once the relevant Markov process has achieved its asymptotic steady state.

In Part 2—as here—I adhere more closely to the game as Parrondo intended it to be played, as a coin-flip-generated walk on the lattice ℤ of positive/negative integers. To establish my methods I look first to Parrondo's A-game.

The A game

The walker (gambler), starting at the origin, advances (wins) with probability x, retreats (loses) with probability X=1-x. After n flips of the loaded coin he stands on one or another of the n + 1 lattice points

$\{-n, -n+2, \ldots, n-2, n\}$

—all of which are even/odd according as n is even/odd (⇒ breaking even is possible iff n is even).

To compute the expected expected results of a 6-flip game we introduce the

2 n + 1 /. n → 6

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13-dimensional Markov matrix

$$\mathbb{A} = \begin{pmatrix} 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & 0 \end{pmatrix};$$

To record our assumption that the walker (gambler) stands initially at the origin (no money on the table) we write

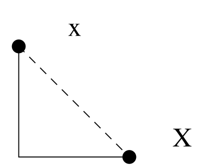
$$P_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

After a single flip we have

```
P1 = A.P0;
P1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x \\ 0 \\ X \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

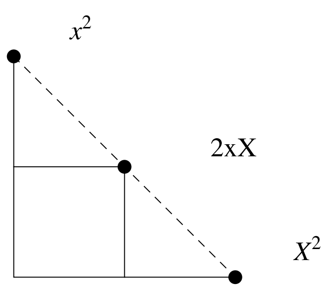
of which the following figure provides diagramatic representation:



After a second flip we have

```
P2 = 𝔸.P1;
P2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x^2 \\ 0 \\ 2\,x\,X \\ 0 \\ X^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



The state vector P_2 is stochastic

```
Total[Transpose[P2]⟦1⟧ /. X → 1 - x] // Simplify
```

1

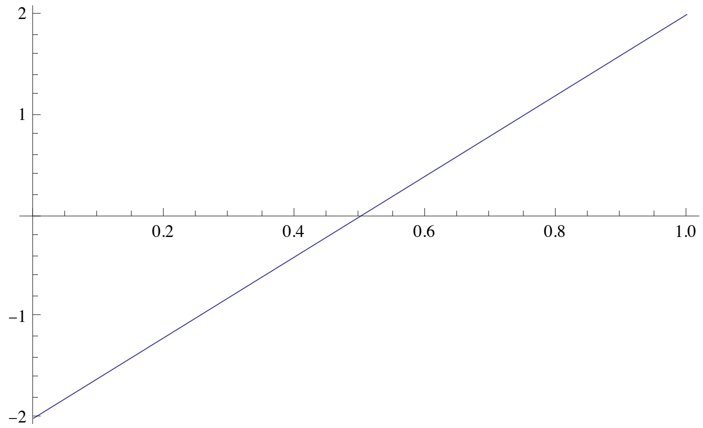
The expected winnings after the 2nd flip are

```
(2) x^2 + (0) 2 x X + (-2) X^2 /. X → 1 - x // Simplify
```

$-2 + 4\,x$

and are positive/negative according as $x \gtrless \frac{1}{2}$:

```
Plot[-2 + 4 x, {x, 0, 1}]
```

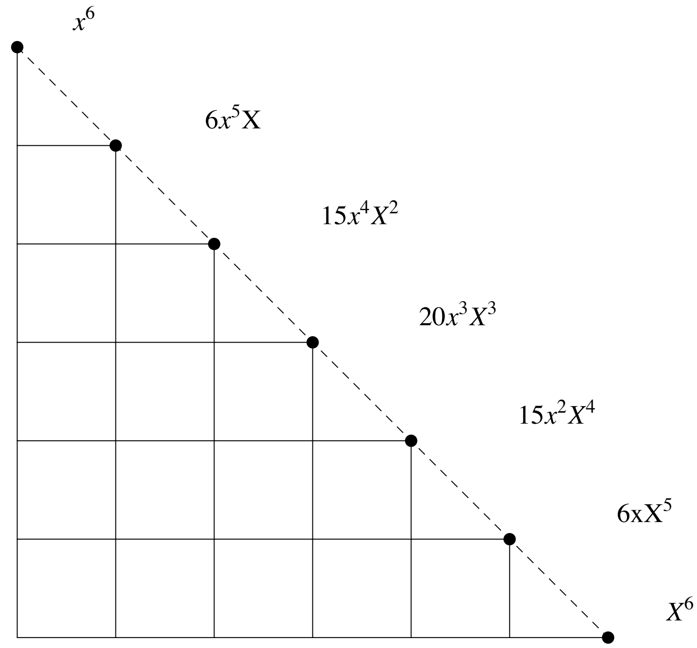


Successive flips produce

```
Table[MatrixForm[MatrixPower[𝔸, n].P0], {n, 0, 6}]
```

$$\left\{\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\x\\0\\X\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\x^2\\0\\2\,x\,X\\0\\X^2\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\x^3\\0\\3\,x^2\,X\\0\\3\,x\,X^2\\0\\X^3\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\x^4\\0\\4\,x^3\,X\\0\\6\,x^2\,X^2\\0\\4\,x\,X^3\\0\\X^4\\0\\0\end{pmatrix},\begin{pmatrix}0\\x^5\\0\\5\,x^4\,X\\0\\10\,x^3\,X^2\\0\\10\,x^2\,X^3\\0\\5\,x\,X^4\\0\\X^5\\0\end{pmatrix},\begin{pmatrix}x^6\\0\\6\,x^5\,X\\0\\15\,x^4\,X^2\\0\\20\,x^3\,X^3\\0\\15\,x^2\,X^4\\0\\6\,x\,X^5\\0\\X^6\end{pmatrix}\right\}$$

so after 6 flips we have the situation shown below:



where the coefficients are recognized to be binomial coefficients:

```
Table[Binomial[6, k], {k, 0, 6}]
```

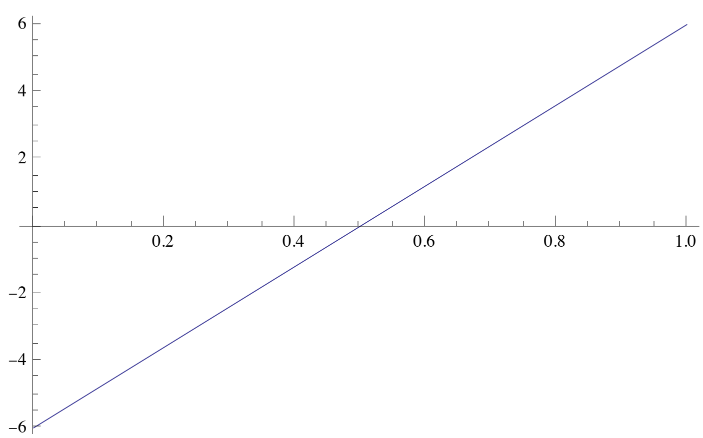
{1, 6, 15, 20, 15, 6, 1}

The expected winnings after the 6th flip are

```
(6) x^6 + (4) 6 x^5 X + (2) 15 x^4 X^2 + (0) 20 x^3 X^3 + (-2) 15 x^2 X^4 + (-4) 6 x X^5 + (-6) X^6 /. X → 1 - x //
 Simplify
```

-6 + 12 x

```
Plot[-6 + 12 x, {x, 0, 1}]
```



Evidently our expected winnings after n flips are given by

$$WA[x\_, n\_] := Simplify\left[\sum_{k=0}^{n} (n - 2 k)\ Binomial[n, k]\ x^{n-k}\ (1 - x)^k\right]$$

```
WA[x, n]
```

n (-1 + 2 x)

This result could hardly be simpler, since it can be formulated

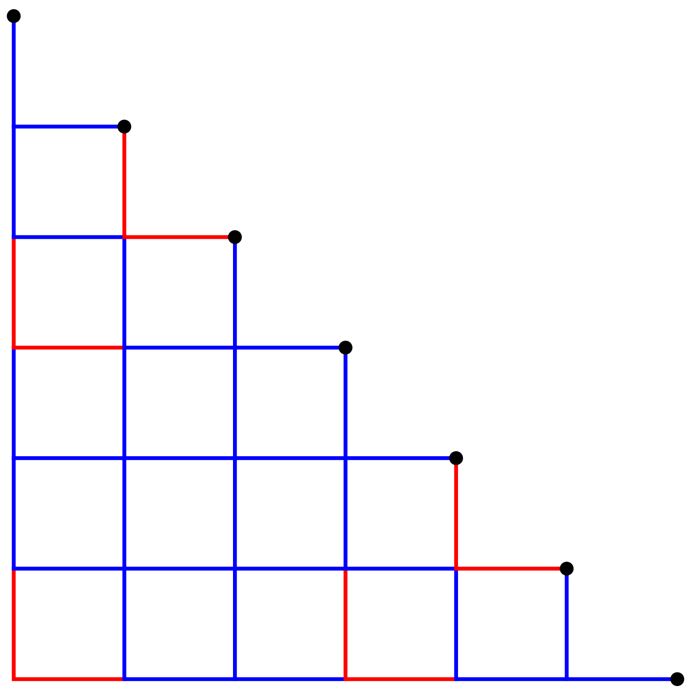
```
WA[x, n] == n WA[x, 1]
```

```
True
```

Players of the A game can expect to win/lose according as $x \gtrless \frac{1}{2}$, and the amount they can expect to win is proportional to the number of flips they play. All of which is almost obvious.

The B game

"Exceptional" steps are controlled by z. They occur at lattice sites that are 0 mod 3, and are indicated by red lines in my figures. "Normal" steps are controlled by y, and are indicted by blue lines.



The analytical theory proceeds from

$$\mathbb{B} = \begin{pmatrix} 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Z & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z & 0 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y & 0 & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z & 0 & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y & 0 \end{pmatrix};$$

Results produced by the first 6 B-flips are shown below:

```
Table[MatrixForm[MatrixPower[𝔹, n].P₀], {n, 0, 6}] // TableForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ z \\ 0 \\ Z \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ y\,z \\ 0 \\ Y\,z + y\,Z \\ 0 \\ Y\,Z \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ y^2\,z \\ 0 \\ y\,Y\,z + z\,(Y\,z + y\,Z) \\ 0 \\ y\,Y\,Z + Z\,(Y\,z + y\,Z) \\ 0 \\ Y^2\,Z \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ y^2 z^2 \\ 0 \\ y z (y Y + y Z) + y z (Y z + y Z) \\ 0 \\ y Y^2 z + y^2 Y Z + (Y z + y Z)^2 \\ 0 \\ Y (y Y + Y z) Z + Y Z (Y z + y Z) \\ 0 \\ Y^2 Z^2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ y^3 z^2 \\ 0 \\ y^2 Y z^2 + y (y z (y Y + y Z) + y z (Y z + y Z)) \\ 0 \\ Y (y z (y Y + y Z) + y z (Y z + y Z)) + z \left(y Y^2 z + y^2 Y Z + (Y z + y Z)^2\right) \\ 0 \\ y (Y (y Y + Y z) Z + Y Z (Y z + y Z)) + Z \left(y Y^2 z + y^2 Y Z + (Y z + y Z)^2\right) \\ 0 \\ y Y^2 Z^2 + Y (Y (y Y + Y z) Z + Y Z (Y z + y Z)) \\ 0 \\ Y^3 Z^2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} y^4 z^2 \\ 0 \\ 2 y^2 z (y Y z + z (Y z + y Z)) \\ 0 \\ \left(y (y Y + Y z) + y^2 Z\right) (y Y z + z (Y z + y Z)) + 2 y^2 z (y Y Z + Z (Y z + y Z)) \\ 0 \\ 2 y^2 Y^2 z Z + (Y (y Y + Y z) + y Y Z) (y Y z + z (Y z + y Z)) + (y Y z + y (y Y + y Z)) (y Y Z + Z (Y z + y Z)) \\ 0 \\ 2 Y^2 Z (y Y z + z (Y z + y Z)) + \left(Y^2 z + Y (y Y + y Z)\right) (y Y Z + Z (Y z + y Z)) \\ 0 \\ 2 Y^2 Z (y Y Z + Z (Y z + y Z)) \\ 0 \\ Y^4 Z^2 \end{pmatrix}$$

Those results are markedly more complicated than their A-game counterparts. But they are—as I verify in the case n = 6—stochastic:

```
MatrixPower[𝔹, 6].P0 /. {Y → 1 - y, Z → 1 - z} MatrixPower[𝔹, 6].P0 /.
   {Y → 1 - y, Z → 1 - z} // Simplify
Total[Transpose[%][[1]]] // Simplify
```

```
{{y^4 z^2}, {0}, {-2 y^2 (y^2 + 2 y (-1 + z) - z) z^2}, {0},
 {y z (y^4 + 6 y^3 (-1 + z) - 6 y (-1 + z) z + z^2 + 2 y^2 (4 - 9 z + 4 z^2))},
 {0}, {y^5 (1 - 2 z) + y (5 - 6 z) z^2 + z^3 + y^4 (-4 + 15 z - 10 z^2) +
   2 y^2 z (5 - 12 z + 6 z^2) + y^3 (4 - 26 z + 32 z^2 - 8 z^3)}, {0}, {(-1 + y)
   (y^4 (-1 + z) + 3 (-1 + z) z^2 + 2 y^3 (2 - 5 z + 3 z^2) - 2 y z (4 - 9 z + 5 z^2) + 4 y^2 (-1 + 5 z - 6 z^2 + 2 z^3))},
 {0}, {-2 (-1 + y)^2 (y^2 + 2 y (-1 + z) - z) (-1 + z)^2}, {0}, {(-1 + y)^4 (-1 + z)^2}}
```

```
1
```

To compute the score (expected winnings) in that case I introduce some notation:

```
Q = Transpose[MatrixPower[𝔹, 6].P0 /. {Y → 1 - y, Z → 1 - z}][[1]]
```

```
{y^4 z^2, 0, 2 y^2 z ((1 - y) y z + z (y (1 - z) + (1 - y) z)), 0,
 2 y^2 z ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)) + (y^2 (1 - z) + y ((1 - y) y + (1 - y) z))
    ((1 - y) y z + z (y (1 - z) + (1 - y) z)), 0, 2 (1 - y)^2 y^2 (1 - z) z +
   (y ((1 - y) y + y (1 - z)) + (1 - y) y z) ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)) +
   ((1 - y) y (1 - z) + (1 - y) ((1 - y) y + (1 - y) z)) ((1 - y) y z + z (y (1 - z) + (1 - y) z)), 0,
 ((1 - y) ((1 - y) y + y (1 - z)) + (1 - y)^2 z) ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)) +
  2 (1 - y)^2 (1 - z) ((1 - y) y z + z (y (1 - z) + (1 - y) z)), 0,
 2 (1 - y)^2 (1 - z) ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)), 0, (1 - y)^4 (1 - z)^2}
```

```
Length[Q]
```

```
13
```

```
WQ = (6) Q[[1]] + (4) Q[[3]] + (2) Q[[5]] + (0) Q[[7]] - 2 Q[[9]] - 4 Q[[11]] - 6 Q[[13]]
```

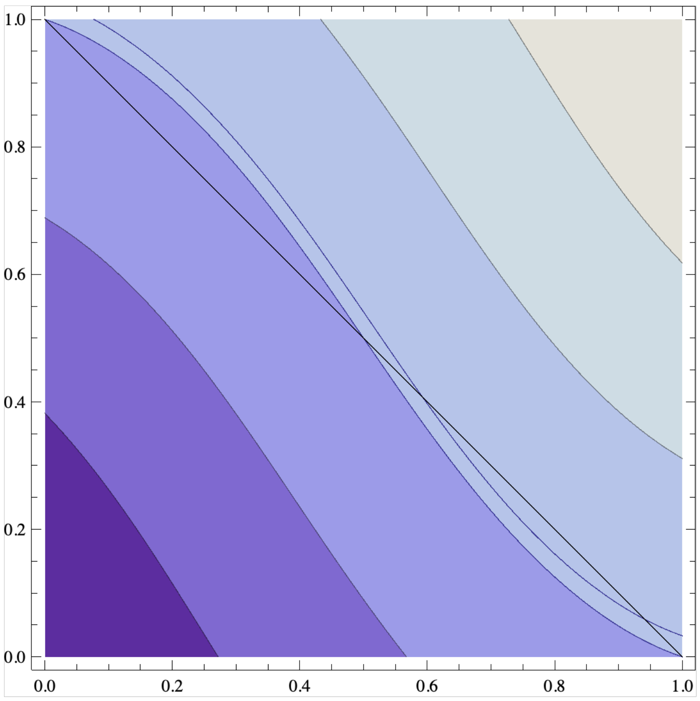
```
-6 (1 - y)^4 (1 - z)^2 + 6 y^4 z^2 - 8 (1 - y)^2 (1 - z) ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)) +
 8 y^2 z ((1 - y) y z + z (y (1 - z) + (1 - y) z)) -
 2 (((1 - y) ((1 - y) y + y (1 - z)) + (1 - y)^2 z) ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)) +
     2 (1 - y)^2 (1 - z) ((1 - y) y z + z (y (1 - z) + (1 - y) z))) +
 2 (2 y^2 z ((1 - y) y (1 - z) + (1 - z) (y (1 - z) + (1 - y) z)) +
     (y^2 (1 - z) + y ((1 - y) y + (1 - y) z)) ((1 - y) y z + z (y (1 - z) + (1 - y) z)))
```

```
WQ // Simplify
```

```
2 (-3 + y^5 + y^3 (4 - 6 z) + 2 z + 2 z^2 - z^3 + y^4 (-4 + 3 z) + y^2 (-2 + 4 z + 6 z^2 - 4 z^3) + y (4 - 7 z^2 + 4 z^3))
```

Looking to the following figure

```
Score6 = ContourPlot[WQ, {y, 0, 1}, {z, 0, 1}];
BreakEven = ContourPlot[WQ == 0, {y, 0, 1}, {z, 0, 1}];
Win = ContourPlot[WQ == .2, {y, 0, 1}, {z, 0, 1}];
ReferenceLine = Graphics[Line[{{0, 1}, {1, 0}}]];
Show[{Score6, BreakEven, Win, ReferenceLine}]
```



we see that the B-game is a winning game when {y, z} sits above the curved diagonal. I introduced a straight diagonal to expose asymmetry (if any) of the curved diagonal: if present at all it is very slight.

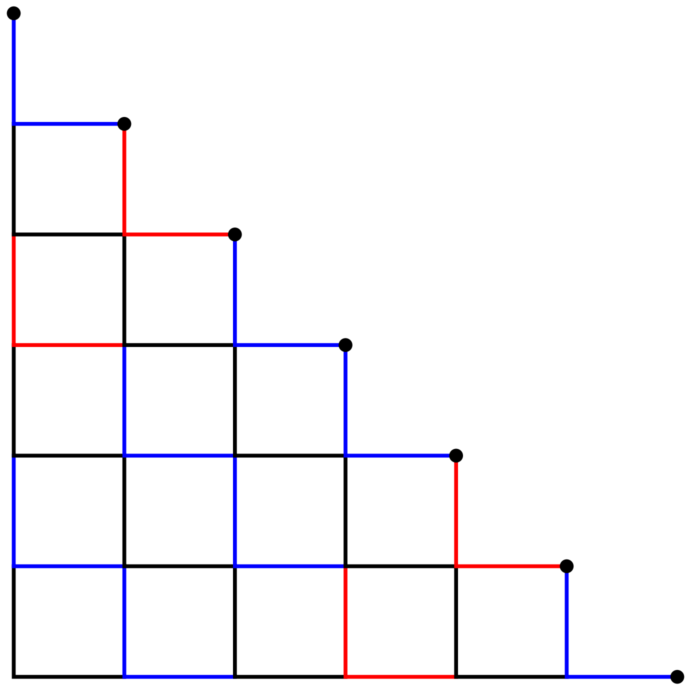
At y = z the B-game becomes a copy of the A-game; all complications evaporate, and we recover

```
WQ /. z → y // Simplify
```

$-6+12\,y$

The ABABAB game

Parrondo's version of the game is played AABBAABBAABB, but I found in Part 3 that the Paradox is conspicuous also in the simpler ABABAB game, which is the version discussed below. I preceed in the assumption (which should make no difference in the long term?) that the A-player makes the first move. The game tree then assumes the following form



For the 6-move game ABABAB we have

```
ℂ = 𝔹.𝔸;
```

```
R = Transpose[MatrixPower[ℂ, 3].P0 /. {X → 1 - x, Y → 1 - y, Z → 1 - z}]〚1〛
```

$$\begin{aligned}
\{&x^3 y^2 z,\ 0,\ x\,(x\,y\,(x\,(1-y)+(1-x)\,y)+x\,y\,((1-x)\,y+x\,(1-z)))\,z+x^2\,y\,z\,(x\,(1-y)+(1-x)\,z),\\
&0,\ x\,y\,\left(2\,(1-x)\,x\,(1-y)\,y+(x\,(1-y)+(1-x)\,y)^2\right)+\\
&\quad(x\,y\,(x\,(1-y)+(1-x)\,y)+x\,y\,((1-x)\,y+x\,(1-z)))\,((1-x)\,y+x\,(1-z))+(1-x)\,x^2\,y\,(1-z)\,z,\\
&0,\ (x\,(1-y)+(1-x)\,y)\,\left(2\,(1-x)\,x\,(1-y)\,y+(x\,(1-y)+(1-x)\,y)^2\right)+\\
&\quad(1-x)\,(1-y)\,(x\,y\,(x\,(1-y)+(1-x)\,y)+x\,y\,((1-x)\,y+x\,(1-z)))+\\
&\quad x\,y\,((1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,y)+(1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,z)),\ 0,\\
&(1-x)\,(1-y)\,\left(2\,(1-x)\,x\,(1-y)\,y+(x\,(1-y)+(1-x)\,y)^2\right)+(1-x)^2\,x\,(1-y)\,(1-z)\,z+\\
&\quad(x\,(1-y)+(1-x)\,z)\,((1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,y)+(1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,z)),\\
&0,\ (1-x)^2\,(1-y)\,((1-x)\,y+x\,(1-z))\,(1-z)+\\
&\quad(1-x)\,(1-z)\,((1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,y)+(1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,z)),\\
&0,\ (1-x)^3\,(1-y)^2\,(1-z)\}
\end{aligned}$$

```
WR = (6) R〚1〛 + (4) R〚3〛 + (2) R〚5〛 + (0) R〚7〛 - 2 R〚9〛 - 4 R〚11〛 - 6 R〚13〛
```

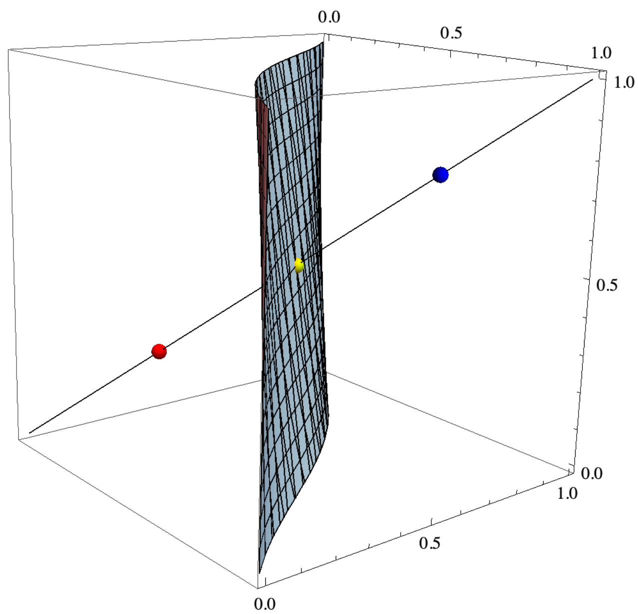
$$\begin{aligned}
&-6\,(1-x)^3\,(1-y)^2\,(1-z)+6\,x^3\,y^2\,z+\\
&2\,\Big(x\,y\,\left(2\,(1-x)\,x\,(1-y)\,y+(x\,(1-y)+(1-x)\,y)^2\right)+(x\,y\,(x\,(1-y)+(1-x)\,y)+\\
&\qquad x\,y\,((1-x)\,y+x\,(1-z)))\,((1-x)\,y+x\,(1-z))+(1-x)\,x^2\,y\,(1-z)\,z\Big)+\\
&4\,\left(x\,(x\,y\,(x\,(1-y)+(1-x)\,y)+x\,y\,((1-x)\,y+x\,(1-z)))\,z+x^2\,y\,z\,(x\,(1-y)+(1-x)\,z)\right)-\\
&4\,\Big((1-x)^2\,(1-y)\,((1-x)\,y+x\,(1-z))\,(1-z)+\\
&\qquad(1-x)\,(1-z)\,((1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,y)+(1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,z))\Big)-\\
&2\,\Big((1-x)\,(1-y)\,\left(2\,(1-x)\,x\,(1-y)\,y+(x\,(1-y)+(1-x)\,y)^2\right)+(1-x)^2\,x\,(1-y)\,(1-z)\,z+\\
&\qquad(x\,(1-y)+(1-x)\,z)\,((1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,y)+(1-x)\,(1-y)\,(x\,(1-y)+(1-x)\,z))\Big)
\end{aligned}$$

```
WR // Simplify
```

$$2\left((-1+y)\left(3+y+y^2-z-z^2\right)+x^3\left(4y^2+(3-2z)z-y(3+2z)\right)+ x^2\left(5y^3+5(-1+z)z-y^2(13+z)+y\left(5+8z-4z^2\right)\right)+ x\left(3-5y^3+z-4z^2+y^2(6+z)+y\left(-1-2z+4z^2\right)\right)\right)$$

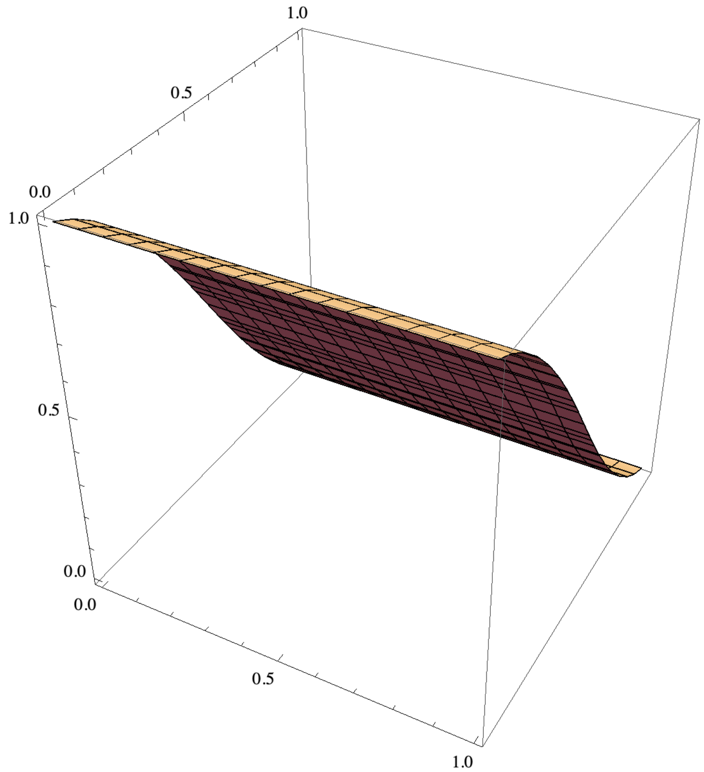
The following figure shows the breakeven surface for the 6-more ABABAB game. The winning region is identified by the blue sphere, the losing region by the red sphere. A yellow sphere marks the center of the parameter space.

```
ABBreakEven6 = ContourPlot3D[WR == 0, {x, 0, 1}, {y, 0, 1}, {z, 0, 1}];
ABGauge = Graphics3D[{
    Line[{{0, 0, 0}, {1, 1, 1}}],
    {Red, Sphere[{0.25, 0.25, 0.25}, 0.02]},
    {Yellow, Sphere[{0.50, 0.50, 0.50}, 0.02]},
    {Blue, Sphere[{0.75, 0.75, 0.75}, 0.02]}
   }];
Show[{ABBreakEven6, ABGauge}]
```

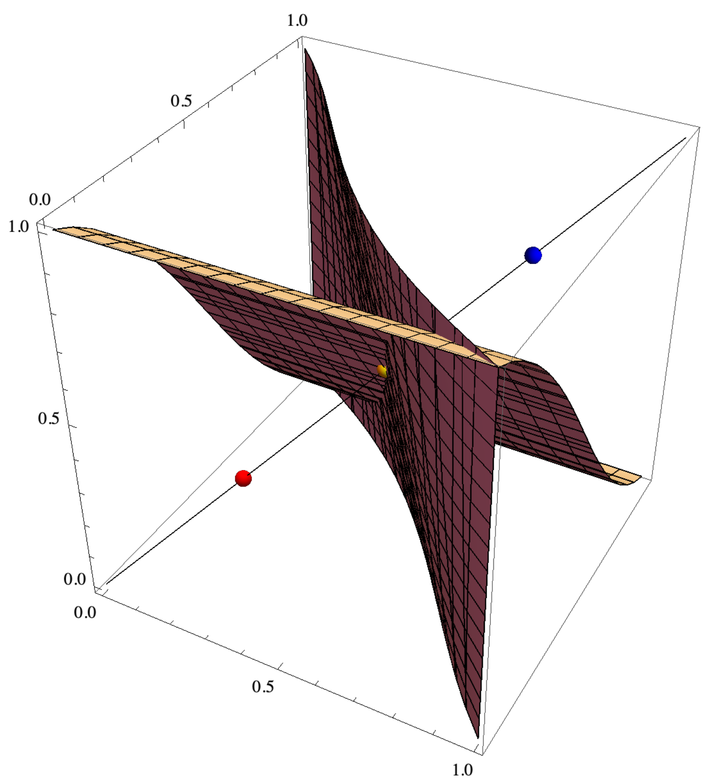


■ Demonstration of the emergence of Parrondo's Paradox

```
BBreakEven6 = ContourPlot3D[WQ == 0, {x, 0, 1}, {y, 0, 1}, {z, 0, 1}]
```

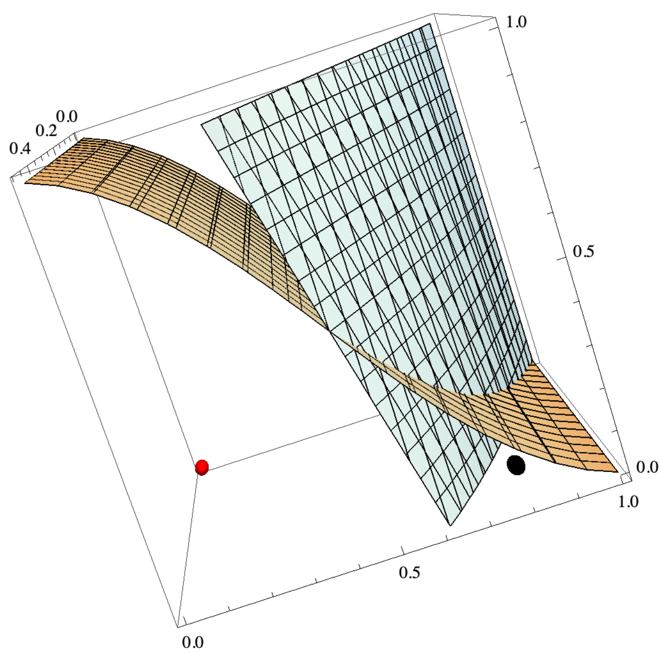



```
Show[{ABBreakEven6, BBreakEven6, ABGauge}]
```



The A-game loses if x < $\frac{1}{2}$. So I construct the fractions of those null surfaces that lie in that semicube. I place a red sphere at the origin (to mark the sector in which A, B and AB all lose) and a black sphere at a point which was seen in preliminary versions of the figure to comprise a plausible Parrondo point:

```
SemiABBreakEven6 = ContourPlot3D[WR == 0, {x, 0, 0.5}, {y, 0, 1}, {z, 0, 1}];
SemiBBreakEven6 = ContourPlot3D[WQ == 0, {x, 0, 0.5}, {y, 0, 1}, {z, 0, 1}];
MarksOrigin = Graphics3D[{Red, Sphere[{0, 0, 0}, 0.02]}];
ParrondoPoint = Graphics3D[{Black, Sphere[{0.45, 0.8, 0.05}, 0.02]}];
Show[{SemiABBreakEven6, SemiBBreakEven6, MarksOrigin, ParrondoPoint},
 AspectRatio → Automatic]
```



Though the game has been short (only three repetitions of AB) it has exposed already Parrondo's paradoxical result: at {x, y, z} = {0.45, 0.80. 0.05} **the A-game and B-game lose, but when played in AB alternation they win**:

```
WA[x, 6] /. x → 0.45
WQ /. {x → 0.45, y → 0.8, z → 0.05}
WR /. {x → 0.45, y → 0.8, z → 0.05}
```

```
-0.6
```

```
-0.41265
```

```
0.722419
```

The "Parrondo points" are seen to live in a little tent-like sector of the parameter space.

To play longer games we would have to construct larger 𝔸 and 𝔹 matrices. For example, to play 10 AB sequences we—since the possible scores in such games range on [-20, +20]—would have to construct 41-dimensional matrices. It would become advantageous at that point to bring into play *Mathematica*'s sparse matrix resources

```
s = SparseArray[{{1, 1} → 1, {2, 2} → 2, {3, 3} → 3, {1, 3} → 4}]
MatrixForm[s]
```

```
SparseArray[<4>, {3, 3}]
```

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and to employ programing techniques of which I have no command.